

Quasipolynomial Solutions to the Hofstadter Q-Recurrence

Nathan Fox*

Abstract

In 1991, Solomon Golomb discovered a quasilinear solution to Hofstadter's Q -recurrence. In this paper, we construct eventual quasipolynomial solutions of all positive degrees to Hofstadter's recurrence.

1 Introduction

In the 1960s, Douglas Hofstadter introduced his Q sequence [3, pp. 137-138]. This sequence is defined by the recurrence $Q(n) = Q(n - Q(n - 1)) + Q(n - Q(n - 2))$ along with the initial conditions $Q(1) = 1$ and $Q(2) = 1$. Sequences defined in this way are often referred to as *meta-Fibonacci sequences* [1]. Though simple to define, this sequence appears to behave unpredictably. To this day, it is even open whether this sequence is defined for all n . It is conceivable that $Q(k) > k$ for some k , in which case $Q(k + 1)$ would be undefined, as calculating it would refer to Q of a nonpositive number. Throughout this paper, though, we will use the convention that $Q(n) = 0$ for $n \leq 0$. (We will call such an occurrence an *underflow*.) This may seem like cheating, but we could just as well replace the existence question about the Q sequence by the equivalent question of whether $Q(n) \leq n$ for all n . Other authors have also used this convention [4].

In 1991, Golomb discovered a more predictable variation of Hofstadter's Q -sequence [2]. He used the same recurrence, but, instead of the initial conditions $Q(1) = 1$ and $Q(2) = 1$, he used initial conditions $Q(1) = 3$, $Q(2) = 2$, and $Q(3) = 1$. This leads to a quasilinear sequence that can be described as follows:

$$\begin{cases} Q(3n) = 3n - 2 \\ Q(3n + 1) = 3 \\ Q(3n + 2) = 3n + 2. \end{cases}$$

Given that one solution like this exists, it is conceivable that other related solutions exist. In particular, under the aforementioned convention, it is plausible that Hofstadter's recurrence

*Department of Mathematics, Rutgers University, Piscataway, New Jersey, fox@math.rutgers.edu

could have solutions where one equally-spaced subsequence grows quadratically. This would occur if, for example, $Q(qn + r)$ equals $Q(qn + r - q)$ plus a linear polynomial, as the sequence $(Q(qn + r))_{n \geq 1}$ would satisfy the recurrence $a_n = a_{n-1} + An + B$ for some A and B .

In this paper, we show that quadratic solutions of this form do exist for Hofstadter's Q -recurrence. In fact, we construct eventually-quasipolynomial solutions to the Q -recurrence of all positive degrees.

2 The Construction

First, we define the following:

Definition 1. Fix integers $d \geq 1$ and $k \geq -1$. Let

$$p_{d,k}(n) = 3d \binom{n+k}{1+k} + \sum_{i=1}^k (3i+2) \binom{n-1+k-i}{k-i}.$$

Observe that $p_{d,k}$ is a polynomial in n of degree $k+1$. In particular, $p_{d,-1} = 3d$, and $p_{d,0} = 3dn$. We will prove the following theorem:

Theorem 1. Fix a degree $d \geq 1$. Define a sequence $(a_m)_{m \geq 1}$ as follows:

$$a_{3dn+r} = \begin{cases} 3d-2 & 3dn+r=1 \\ 0 & 3dn+r=2 \\ p_{d,\frac{r}{3}}(n) & r \equiv 0 \pmod{3} \\ 3d & r \equiv 1 \pmod{3} \text{ and } 3dn+r > 2 \\ 3 & r \equiv 2 \pmod{3} \text{ and } r \neq 3d-1 \text{ and } 3dn+r > 2 \\ 2 & r = 3d-1 \text{ and } 3dn+r > 2, \end{cases}$$

where $0 \leq r < 3d$ always. Then, (a_m) satisfies the recurrence $Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2))$ after an initial condition of length $3d+2$.

We will use the following lemmas:

Lemma 1. For all integers $d \geq 1$ and $k \geq 0$ we have $p_{d,k}(n) = p_{d,k-1}(n) + p_{d,k}(n-1)$.

Proof. We have

$$\begin{aligned}
p_{d,k-1}(n) + p_{d,k}(n-1) &= 3d \binom{n+k-1}{k} + \sum_{i=1}^{k-1} (3i+2) \binom{n-2+k-i}{k-i-1} \\
&\quad + 3d \binom{n+k-1}{1+k} + \sum_{i=1}^k (3i+2) \binom{n-2+k-i}{k-i} \\
&= 3d \left(\binom{n+k-1}{k} + \binom{n+k-1}{1+k} \right) \\
&\quad + \sum_{i=1}^{k-1} (3i+2) \left(\binom{n-2+k-i}{k-i-1} + \binom{n-2+k-i}{k-i} \right) \\
&\quad + (3k+2) \binom{n-2}{0}.
\end{aligned}$$

Applying Pascal's Identity yields

$$\begin{aligned}
p_{d,k-1}(n) + p_{d,k}(n-1) &= 3d \binom{n+k}{1+k} + \sum_{i=1}^{k-1} (3i-2) \binom{n-1+k-i}{k-i} + (3k-2) \\
&= 3d \binom{n+k}{1+k} + \sum_{i=1}^k (3i-2) \binom{n-1+k-i}{k-i} \\
&= p_{d,k}(n),
\end{aligned}$$

as required. □

Lemma 2. *For all integers $d \geq 1$, $k \geq 1$, and $n \geq 0$ we have*

$$p_{d,k}(n) \geq 3dn + 3k + 2.$$

Proof. First, we observe that

$$p_{d,k}(0) = 3d \binom{k}{1+k} + \sum_{i=1}^k (3i+2) \binom{k-i-1}{k-i}.$$

All of these binomial coefficients are zero, except when $i = k$, since $\binom{-1}{0} = 1$. So, $p_{d,k}(0) = 3k + 2$. This equals $3dn + 3k + 2$, and hence is greater than or equal to it, as required.

Now,

$$\begin{aligned}
p_{d,k}(1) &= 3d \binom{1+k}{1+k} + \sum_{i=1}^k (3i+2) \binom{k-i}{k-i} \\
&= 3d + \sum_{i=1}^k (3i+2) \\
&= 3d + 3 \left(\frac{k^2+k}{2} \right) + 2k \\
&= \frac{3}{2}k^2 + \frac{7}{2}k + 3d.
\end{aligned}$$

So,

$$\begin{aligned}
p_{d,k}(1) - 3d \cdot 1 + 3k + 2 &= \frac{3}{2}k^2 + \frac{7}{2}k + 3d - 3d - 3k - 2 \\
&= \frac{3}{2}k^2 + \frac{1}{2}k - 2 \\
&= \frac{(3k+4)(k-1)}{2}.
\end{aligned}$$

This is greater than or equal to 0, since $k \geq 1$. So, $p_{d,k}(1) \geq 3d + 3k + 2$, as required.

Now, observe that $p_{d,k}$ has nonnegative coefficients, so it is convex. We have seen that its average slope on the interval $[0, 1]$ is at least $3d$, so its derivative for $n > 1$ must be strictly larger than $3d$ everywhere. Therefore, since $p_{d,k}(1) \geq 3d + 3k + 2$, we can conclude that $p_{d,k}(n) \geq 3dn + 3k + 2$ for all $n \geq 0$. \square

We will now prove Theorem 1.

Proof. We will check the three congruence classes mod 3 separately for $m > 3d + 2$. As usual, $m = 3dn + r$ for $0 \leq r < 3d$. We will proceed by induction, so in each case we will assume that all previous values of the sequence are what they should be. Also, in all cases, since $m > 3d + 2$, $m - 3d > 2$. (This will come up when deciding whether or not we are in the special initial conditions for the first two values.)

$r \equiv 0 \pmod{3}$: Assume $r \equiv 0 \pmod{3}$. Then, $m = 3dn + r$ for some n . For convenience, let $\ell = \frac{r}{3}$. We wish to show that $Q(3dn + r) = p_{d,\ell}(n)$. Let $c = 2$ if $r = 0$; otherwise, let $c = 3$. We have,

$$\begin{aligned}
Q(3dn + r) &= Q(3dn + r - Q(3dn + r - 1)) + Q(3dn + r - Q(3dn + r - 2)) \\
&= Q(3dn + r - c) + Q(3dn + r - 3d) \\
&= Q(3dn + r - c) + Q(3d(n - 1) + r) \\
&= Q(3dn + r - c) + p_{d,\ell}(n - 1).
\end{aligned}$$

If $r = 0$, then $\ell = 0$ and

$$Q(3dn + r - c) = Q(3dn + r - 2) = 3d = p_{d,\ell-1}(n).$$

If $r \neq 0$, then $\ell > 0$ and

$$Q(3dn + r - c) = Q(3dn + r - 3) = p_{d,\ell-1}(n).$$

In either case, we have

$$Q(3dn + r) = p_{d,\ell-1}(n) + p_{d,\ell}(n - 1).$$

By Lemma 1, this equals $p_{d,\ell}(n)$, as required.

$r \equiv 1 \pmod{3}$: Assume $r \equiv 1 \pmod{3}$. Then, $m = 3dn + r$ for some n . We wish to show that $Q(3dn + r) = 3d$. For convenience, let $\ell = \frac{r-1}{3}$. We have,

$$\begin{aligned} Q(3dn + r) &= Q(3dn + r - Q(3dn + r - 1)) + Q(3dn + r - Q(3dn + r - 2)) \\ &= Q(3dn + r - p_{d,\ell}(n)) + Q(3dn + r - Q(3dn + r - 2)). \end{aligned}$$

If $\ell = 0$, then $p_{d,\ell}(n) = 3dn$ and $r = 1$. So, in that case, $3dn + r - p_{d,\ell}(n) = r = 1$. Also, in that case $Q(3dn + r - 2) = 2$, so

$$Q(3dn + r - Q(3dn + r - 2)) = Q(3dn + r - 2) = 2.$$

Since $Q(1) = 3d - 2$, we obtain $Q(3dn + r) = 3d - 2 + 2 = 3d$ in the case when $r = 1$.

Otherwise, we have $\ell \geq 1$. In that case, $p_{d,\ell}(n) \geq 3dn + 3\ell + 2$ by Lemma 2. But, $3\ell + 2 = r + 1$ so, $3dn + r - p_{d-1}(n) \leq -1$. This causes the first term to underflow, so $Q(3dn + r - p_{d,\ell}(n)) = 0$. Hence, $Q(3dn + r) = Q(3dn + r - Q(3dn + r - 2))$. In this case, we know $r \neq 1$, so $Q(3dn + r - 2) = 3$. This means that

$$Q(3dn + r - Q(3dn + r - 2)) = Q(3dn + r - 3) = 3d.$$

So, $Q(3dn + r) = 3d$, as required.

$r \equiv 2 \pmod{3}$: Assume $r \equiv 2 \pmod{3}$. Then, $m = 3dn + r$ for some n . Let $c = 2$ if $r = 3d - 1$; otherwise, let $c = 3$. We wish to show that $Q(3dn + r) = c$. For convenience, let $\ell = \frac{r-2}{3}$. We have,

$$\begin{aligned} Q(3dn + r) &= Q(3dn + r - Q(3dn + r - 1)) + Q(3dn + r - Q(3dn + r - 2)) \\ &= Q(3dn + r - 3d) + Q(3dn + r - p_{d,\ell}(n)) \\ &= Q(3d(n - 1) + r) + Q(3dn + r - p_{d,\ell}(n)) \\ &= c + Q(3dn + r - p_{d,\ell}(n)). \end{aligned}$$

If $\ell = 0$, then $p_{d,\ell}(n) = 3dn$ and $r = 2$. So, in that case, $3dn + r - p_{d,\ell}(n) = r = 2$. Since $Q(2) = 0$, we obtain $Q(3dn + r) = c$ in the case when $r = 2$.

Otherwise, we have $\ell \geq 1$. In that case, $p_{d,\ell}(n) \geq 3dn + 3\ell + 2$ by Lemma 2. But, $3\ell + 2 = r$ so, $3dn + r - p_{d-1}(n) \leq 0$, an underflow in the second term. This implies that $Q(3dn + r - p_{d,\ell}(n)) = 0$, so $Q(3dn + r) = c$, as required.

□

Note that the only place we used the $3i + 2$ in the definition of $p_{d,k}(n)$ was to obtain the lower bound of $r + 2$ on the polynomials. So, $3i + 2$ could be replaced by any larger expression, and the proof would still go through. Also, observe that this construction is not a direct generalization of Golomb's construction, as the $d = 1$ case has two constant pieces and one linear piece, unlike Golomb's, which has one constant piece and two linear pieces. Also, Golomb's example is *purely* quasilinear, whereas our $d = 1$ example is only eventually quasilinear. It is unknown whether there exist purely quasipolynomial solutions to the Hofstadter Q -recurrence of degrees greater than 1.

2.1 An Example

As an example of our construction, we will construct a solution to Hofstadter's recurrence with a cubic subsequence. To do this, we set $d = 3$, which means that the sequence values will depend on the congruence class mod 9 of the index. We observe that

$$\begin{aligned}
 p_{3,0} &= 9n \\
 p_{3,1} &= 9 \binom{n+1}{2} + 5 \binom{n-1}{0} = \frac{9}{2}n(n+1) + 5 \\
 &= \frac{9}{2}n^2 + \frac{9}{2}n + 5 \\
 p_{3,2} &= 9 \binom{n+2}{3} + 5 \binom{n}{1} + 8 \binom{n-1}{0} = \frac{9}{6}n(n+1)(n+2) + 5n + 8 \\
 &= \frac{3}{2}n^3 + \frac{9}{2}n^2 + 8n + 8.
 \end{aligned}$$

So, our sequence is defined by $a_1 = 7$, $a_2 = 0$, and for $9n + r > 2$,

$$a_{9n+r} = \begin{cases} 9n & r = 0 \\ 9 & r = 1 \\ 3 & r = 2 \\ \frac{9}{2}n^2 + \frac{9}{2}n + 5 & r = 3 \\ 9 & r = 4 \\ 3 & r = 5 \\ \frac{3}{2}n^3 + \frac{9}{2}n^2 + 8n + 8 & r = 6 \\ 9 & r = 7 \\ 2 & r = 8. \end{cases}$$

After the initial condition 7, 0, 5, 9, 3, 8, 9, 2, 9, 9, 3, repeated applications of the Hofstadter Q -recurrence produce the sequence

$$7, 0, 5, 9, 3, 8, 9, 2, 9, 9, 3, 14, 9, 3, 22, 9, 2, 18, 9, 3, 32, 9, 3, 54, 9, 2, 27, 9, 3, 59, 9, 3, 113, 9, 2, \dots$$

References

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